On Linear Transformations which Leave an Hermitian Form Invariant.

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Any Hermitian form whose determinant is not zero may be reduced by a linear transformation to

$$H = \sum_{i=1}^{n} \lambda_i z_i \bar{z}_i , \tag{1}$$

in which the coefficients λ_i are real and p of them are positive. The number p is invariant for any given form.[*] This fact will be referred to as the *Law of Inertia*.

It is the object of the present paper to consider some of the properties of linear transformations for which H is invariant.[†] For this purpose it is sufficient to consider the cases in which p does not exceed $\frac{1}{2}n$, since the forms H and $-H$ are invariant for the same substitutions.

§ 1. The Transformations S and their Classification.

Let the equations

$$(S) \qquad z_k' = \sum_{i=1}^{n} c_{ki} z_i \qquad (k = 1, 2, \ldots, n)$$

define any linear transformation whose determinant is not zero. If H is unaltered by S, the coefficients c_{ki} must satisfy the conditions

$$\sum_{k=1}^{n} \lambda_k c_{ki} \bar{c}_{kj} = \begin{cases} \lambda_i, & \text{when } j = i, \\ 0, & \text{``} \quad j \neq i. \end{cases} \tag{2}$$

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[*] See A. Loewy: "Ueber bilineare Formen mit conjugirt imaginären Variablen," *Mathematische Annalen*, Vol. 50 (1898), p. 563.

[†] We follow the methods used by Picard for the case $n = 3$ in his memoir, "Sur les formes quadratiques ternaires indéfinies à indeterminées conjuguées et sur les fonctions hyperfuchsiennes correspondantes," *Acta Mathematica*, Vol. V (1884), § 16, rather than the symbolic methods of Frobenius as employed by Loewy, *loc. cit.*

If the n equations (S) be multiplied by $\lambda_k \bar{c}_{kj} (k = 1, 2, \ldots, n)$ and added, we obtain, after using (2), the inverse transformation S^{-1} in the form

$$(S^{-1}) \qquad z_i = \frac{1}{\lambda_i} \sum_{k=1}^{n} \lambda_k \bar{c}_{ki} z_k' \qquad (i = 1, 2, \ldots, n).$$

Since S^{-1} also leaves H invariant its coefficients satisfy the conditions

$$\sum_{i=1}^{n} \frac{1}{\lambda_i} c_{li} \bar{c}_{ki} = \begin{cases} \lambda_k^{-1}, & \text{when } l = k, \\ 0, & \text{``} \quad l \neq k, \end{cases} \qquad (2')$$

which are equivalent to (2).

In what follows we shall not distinguish between two substitutions whose corresponding coefficients differ by a common factor. Such a factor is necessarily of the form $e^{i\phi}$.

The transformation S contains $2n^2$ real constants which are subjected to n^2 real conditions (2). Accordingly, *the substitution S contains n^2 arbitrary real parameters.*[*]

For function-theory purposes it is often desirable to introduce non-homogeneous variables, which will be defined by the equations

$$u_\alpha = z_\alpha / z_n \qquad (\alpha = 1, 2, \ldots, n-1). \qquad (3)$$

Let u_α be separated into real and imaginary parts, $u_\alpha = u_\alpha' + i u_\alpha''$. The values of these $2(n-1)$ variables will be associated with the points of a flat space R of $2(n-1)$ dimensions.

Denote by F the function $H/z_n \bar{z}_n$ of $u_1, \bar{u}_1, \ldots, u_{n-1}, \bar{u}_{n-1}$. The locus

$$F = 0 \qquad (4)$$

is transformed into itself by the substitution S, while any point not on (4) is transformed into a point on the same side of this locus unless $p = \frac{1}{2} n$. We will call the region defined by the inequality $F > 0$ the inside of F.

§ 2. *The Fixed Points.*

We now proceed to the consideration of the points of R which remain fixed for the transformation S. For convenience we will return to the homogeneous notation. At a fixed point we have

$$z_k' = \rho z_k \qquad (k = 1, 2, \ldots, n).$$

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[*] The transformation may be put in a form in which the superfluous parameters are eliminated from the coefficients. See the foot-note to page 200. From the definition of T there given it is readily found to contain n^2 arbitrary real parameters.

Substituting in the equations (S) we obtain the conditions

$$c_{k1} z_1 + c_{k2} z_2 + \cdots + (c_{kk} - \rho) z_k + \cdots + c_{kn} z_n = 0 \ (k = 1, 2, \ldots, n). \quad (5)$$

If C denote the determinant of these equations, then, as is well known, the value of ρ is found from the characteristic equation $C = 0$.

If, on the other hand, we substitute $z' = \rho z$ in the equations (S^{-1}) and divide each one by ρ, we obtain

$$\lambda_1 \bar{c}_{1i} z_1 + \lambda_2 \bar{c}_{2i} z_2 + \cdots + \lambda_i (\bar{c}_{ii} - \rho^{-1}) z_i + \cdots + \lambda_n \bar{c}_{ni} z_n = 0 \ (i = 1, 2, \ldots, n).$$

Denoting the determinant of these equations by C', we obtain for ρ the equation $C' = 0$. After dividing out $\lambda_1 \lambda_2 \ldots \lambda_n$, we observe that if in the resulting equation every letter is replaced by its conjugate, the equation so obtained is the same as $C = 0$ except that ρ is replaced by $1/\bar{\rho}$. Hence, *if ρ is a root of $C = 0$, then $1/\bar{\rho}$ is also a root of the same equation.*[*]

We therefore have a number q of pairs of roots ρ_1, $\rho_2 = 1/\bar{\rho}_1$, ρ_3, $\rho_4 = 1/\bar{\rho}_3$, $\ldots$, ρ_{2q-1}, $\rho_{2q} = 1/\bar{\rho}_{2q-1}$ whose absolute values are reciprocals, and different from unity. The remaining roots ρ_{2q+1}, $\ldots$, ρ_n are then unimodular. Suppose these n roots are all distinct. By a linear transformation T the substitution S may be put in the form $\zeta'_k = \rho_k \zeta_k \ (k = 1, 2, \ldots, n)$. Expressing H in terms of the variables ζ by means of T it takes the form

$$H = \sum_{i=1}^{q} (a_i \zeta_{2i-1} \bar{\zeta}_{2i} + \bar{a}_i \zeta_{2i} \bar{\zeta}_{2i-1}) + \sum_{\gamma=2q+1}^{n} a_\gamma \zeta_\gamma \bar{\zeta}_\gamma.$$

Now make the transformation

$$\zeta_{2i-1} = (\eta_{2i-1} + \eta_{2i})/a_i, \quad \zeta_{2i} = \tfrac{1}{2} (\eta_{2i-1} - \eta_{2i}), \quad \zeta_\gamma = \eta_\gamma,$$

and we obtain

$$H = \sum_{i=1}^{q} (\eta_{2i-1} \bar{\eta}_{2i-1} - \eta_{2i} \bar{\eta}_{2i}) + \sum_{\gamma=2q+1}^{n} a_\gamma \eta_\gamma \bar{\eta}_\gamma. \quad (6)$$

According to the Law of Inertia q can not exceed p. It follows that when the roots of $C = 0$ are all distinct the substitution S belongs to one of $p + 1$ different types according to the value of q. When $q = 0$, S will be called *elliptic*. In other cases it will be called *hyperbolic of the q-th type* ($q = 1, 2, \ldots, p$).

Our next problem is to determine the positions of the fixed points of S. Suppose at first that the roots of $C = 0$ are all distinct. In that case we have

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[*] This result is obtained by Loewy (*loc. cit.*, p. 566) by an application of the symbolic methods and results of Frobenius to the Hermitian form. For the case $n = 3$ it was first given by Picard, *Acta Mathematica*, Vol. V (1884), p. 164.

n distinct fixed points P_1, P_2, ..., P_n corresponding respectively to the values ρ_1, ρ_2, ..., ρ_n of ρ. At the fixed points P_k the equations (S) reduce to $z_i' = \rho_k z_i (i = 1, 2, ..., n)$. If H' denote the transformed form $\Sigma \lambda_i z_i' \bar{z}_i'$, then at P_k we have the relation $H' = \rho_k \bar{\rho}_k H = H$, whence $(\rho_k \bar{\rho}_k - 1) H = 0$. If $|\rho_k|$ is not 1, it follows that H is 0. Hence, 2q *fixed points of a hyperbolic substitution of the q-th type are on the locus F* = 0.

When the roots ρ_k are all distinct, H may be reduced to the form (6) and therefore the coordinates of the fixed point P_γ $(\gamma = 2q + 1, ..., n)$ are $\eta_k = 0 (k \neq \gamma)$. If these be substituted in H, it reduces to $a_\gamma \eta_\gamma \bar{\eta}_\gamma$ and will therefore be positive or negative according to the sign of a_γ. On account of the invariance of the number p and since (6) has already q positive terms, it follows that $p - q$ *of the n* $- 2q$ *remaining fixed points P_γ are inside F* = 0 *and n* $- p - q$ *are outside.* Applying this result to the case $q = 0$ we may say that p *fixed points of an elliptic substitution are inside, and n* $- p$ *are outside the invariant locus F.*

The case of fixed points when the characteristic equation has equal roots will be considered later.

§ 3. *Reduction of H to the Canonical Form. General Case.*

Before taking up the case of equal roots ρ_k, we will first give some general transformations relative to the reduction of any Hermitian form to a canonical form. Let H be written in the most general form $\Sigma a_{ik} \zeta_i \bar{\zeta}_k$, assuming the determinant A of the coefficients to be different from zero. If for a particular coefficient a_{mm} of the main diagonal of A the corresponding minor A_{mm} is not zero, then a first step in the reduction of H to the canonical form is made by means of the transformation

$$\begin{aligned} \zeta_\kappa &= \eta_\kappa + (A_{\kappa m}/A_{mm}) \eta_m \qquad (\varkappa \neq m, \varkappa = 1, ..., n) \\ \zeta_m &= \eta_m . \end{aligned} \left.\right\} \quad (I)$$

Expressed in terms of the new variables H becomes $H_1 + (A/A_{mm}) \eta_m \bar{\eta}_m$, in which $H_1 = \underset{\kappa,\lambda}{\Sigma} a_{\kappa\lambda} \eta_\kappa \bar{\eta}_\lambda$, the indices $\varkappa, \lambda$ taking all integral values from 1 to n with the exception of m. The determinant of the form H_1 is A_{mm}, which is not zero by hypothesis. If any minor of A_{mm} corresponding to an element of its main diagonal is not zero, we make another substitution of type I, and thus continue as long as possible. If a stage is reached at which (I) can no longer be used, we employ a transformation of the following type.

Suppose that in A the minor A_{mm} is zero but that the minor A_{1m} is not zero. In that case we can apply the transformation *

$$\left.\begin{array}{c} \zeta_\lambda = \eta_\lambda + (A_{\lambda m}/A_{1m})\,\zeta_1, \\ (A/A_{1m})\,\zeta_1 = \tfrac{1}{2}\left[(1-a_{mm})\,\eta_1 + (1+a_{mm})\,\eta_m\right] - \sum_\lambda a_{\lambda m}\,\eta_\lambda, \\ \zeta_m = \eta_1 - \eta_m, \end{array}\right\} \quad \text{(II)}$$

in which λ takes all values from 2 to n with the exception of m. The form H reduces to

$$\eta_1\bar{\eta}_1 - \eta_m\bar{\eta}_m + \sum_{\lambda,\,\mu} a_{\lambda\mu}\,\eta_\lambda\,\bar{\eta}_\mu \qquad (\lambda,\,\mu = 2,\,3,\,\ldots,\,m-1,\,m+1,\,\ldots,\,n).$$

For, from the first equation (II) we obtain by changing to conjugate imaginary values, multiplying by $a_{i\lambda}$, and summing with respect to λ,

$$a_{i1}\,\breve{\zeta}_1 + \sum_\lambda a_{i\lambda}\,\breve{\zeta}_\lambda = a_{i1}\,\breve{\zeta}_1 + \sum_\lambda a_{i\lambda}\,[\bar{\eta}_\lambda + (\bar{A}_{\lambda m}/\bar{A}_{1m})\,\breve{\zeta}_1]. \tag{7}$$

Since $a_{i\lambda} = \bar{a}_{\lambda i}$, $\bar{A} = A$, and $A_{mm} = 0$, the terms in the right member which contain $\breve{\zeta}_1$ vanish unless $i = m$, in which case they reduce to $(A/\bar{A}_{1m})\,\breve{\zeta}_1$ and we have

$$a_{m1}\,\breve{\zeta}_1 + \sum_\lambda a_{m\lambda}\,\breve{\zeta}_\lambda = \sum_\lambda a_{m\lambda}\,\bar{\eta}_\lambda + (A/\bar{A}_{1m})\,\breve{\zeta}_1. \tag{7'}$$

Multiply (7) by ζ_i and sum with respect to i for all values except m. This gives

$$\sum_{\alpha,\,\beta} a_{\alpha\beta}\,\zeta_\alpha\,\breve{\zeta}_\beta = \sum_{\lambda,\,\mu} a_{\lambda\mu}\,\eta_\lambda\,\bar{\eta}_\mu, \tag{8}$$

in which $\alpha,\,\beta$ take all values except m, and $\lambda,\,\mu$ all values except 1 and m. Now write H in the form

$$\sum a_{\alpha\beta}\,\zeta_\alpha\,\breve{\zeta}_\beta + \sum a_{\alpha m}\,\zeta_\alpha\,\breve{\zeta}_m + \sum a_{m\alpha}\,\zeta_m\,\breve{\zeta}_\alpha + a_{mm}\,\zeta_m\,\breve{\zeta}_m$$

and use the second and third equations (II) together with the relations (7') and (8). This leads to the result previously indicated.

§4. *Reduction of H when C Has Equal Roots.*

Suppose that the characteristic equation $C = 0$ has a number of equal roots $\rho_1 = \rho_2 = \ldots = \rho_m = \rho$. There are two cases to be considered, according as ρ is unimodular or not.

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* This, and (I), are adapted from Gundelfinger, "Ueber die Transformation einer quadratischen Form in eine Summe von Quadraten," *Crelle*, Vol. XCI (1881), p. 223.

In the first case we suppose S reduced to the normal form

$$\begin{vmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & M_{\mu+2} \end{vmatrix} \qquad (9)$$

in which the M_i are square matrices defined as follows:

$$M_i = \begin{vmatrix} \rho & \rho & 0 & \cdots & 0 \\ 0 & \rho & \rho & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \rho & \rho \\ 0 & \cdots & \cdots & 0 & \rho \end{vmatrix} (i = 1, \cdots, \mu), \ M_{\mu+1} = \begin{vmatrix} \rho & 0 & \cdots & 0 \\ 0 & \rho & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \rho \end{vmatrix}.$$

It is not necessary to specify the form of $M_{\mu+2}$, except that it does not depend on ρ. Denote the orders of these matrices by $e_1, e_2, \cdots, e_\mu, e_{\mu+1}$ respectively. The integers $e_1, \cdots, e_\mu$ are the exponents of those elementary divisors of C containing the root ρ which are greater than unity. We suppose them arranged in decreasing order of magnitude, $e_1 \geqq e_2 \geqq e_3 \cdots \geqq e_\mu \geqq 2$.

The condition that H be invariant for S may be expressed by the matrix formula

$$S' A \bar{S} = A, \qquad (10)$$

in which A is the matrix of the coefficients a_{ik} of H, $\bar{S}$ is the matrix obtained from S by changing every element into its conjugate value, and S' is obtained from S by interchanging rows and columns. In forming the product the elements of the i-th row of S' are multiplied into the elements of the k-th column of A to form the element of the i-th row and k-th column of the product matrix. The resulting matrix is combined in a similar manner with $\bar{S}$. That (10) is the condition for the invariance of H may readily be seen by substituting $z_i' = \sum\limits_\lambda c_{i\lambda} z_\lambda$, $\bar{z}_k' = \sum \bar{c}_{k\mu} \bar{z}_\mu$ in $H' = \sum a_{ik} z_i' \bar{z}_k'$ and making the matrix of the result identical with A.*

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* The substitution S can be expressed in but one way in the form $(\bar{A} + \bar{T})^{-1} (\bar{A} - \bar{T})$ in which T is subject to the restriction $T + \bar{T}' = 0$. The sum $A + T$ means the matrix whose elements are the sums of the corresponding elements in A and T. It may readily be shown that any substitution of this form changes H into itself. For, the matrix equation (or identity)

$$\bar{A}' + \bar{T}' + T + TA^{-1} \bar{T}' = \bar{A}' - \bar{T}' - T + TA^{-1} \bar{T}'$$

may (by use of $\bar{A}' = A$, the condition that H is an Hermitian form) be written

$$(1 + TA^{-1}) (\bar{A}' + \bar{T}') = (1 - TA^{-1}) (\bar{A}' - \bar{T}')$$

or

$$\bar{U} A^{-1} U' = \bar{V} A^{-1} V',$$

Using the form (9) for S, the product (10) is

$$\begin{vmatrix} S_{11}, & S_{12}, & \ldots, & S_{1,\,\mu+2} \\ S_{21}, & S_{22}, & \ldots, & S_{2,\,\mu+2} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ S_{\mu+2,1}, & S_{\mu+2,2}, & \ldots, & S_{\mu+2,\,\mu+2} \end{vmatrix}, \tag{11}$$

in which S_{ik} represents a matrix of e_i rows and e_k columns constructed in the following manner. Let $\mathbf{M}'_i$ denote the matrix obtained from S' by replacing all its elements by zeros except those in M'_i, and let $\overline{\mathbf{M}}_k$ be formed similarly from $\overline{S}$. Then construct the product $\mathbf{M}'_i A \overline{\mathbf{M}}_k$ in the same way as described for (10). The non-zero elements of this product form a matrix of e_i rows and e_k columns which is denoted by S_{ik}. If now we impose the condition that the matrix (11) is identical with A, the matrices S_{ik} take the special forms

$$\begin{vmatrix} 0 & 0 & \ldots\ldots\ldots & 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 & \ldots\ldots\ldots & \ldots\ldots & 0 \quad a_{a+r_i,\,-r_k,\,\beta+r_k} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 & 0 & a_{a+r_i-2,\,\beta+2} & \ldots\ldots \\ 0 & a_{a+r_i-1,\,\beta+1} & \ldots\ldots & \ldots\ldots \\ a_{a+r_i,\,\beta} & a_{a+r_i,\,\beta+1} & \ldots\ldots & a_{a+r_i,\,\beta+r_k} \end{vmatrix}$$

in which $\alpha = e_1 + \ldots + e_{i-1}$, $\beta = e_1 + \ldots + e_{k-1}$, $r_i = e_i - 1$, $r_k = e_k - 1$, and i, k do not exceed μ, $i < k$;

$$S_{ki} = \begin{vmatrix} 0 & \ldots\ldots\ldots\ldots\ldots\ldots\ldots & 0 \quad a_{a,\,\beta+r_k} \\ 0 & \ldots\ldots\ldots\ldots & 0 \quad a_{a+1,\,\beta+r_k-1} & a_{a+1,\,\beta+r} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 & \ldots\ldots & 0 \quad a_{a+r_i,\,\beta+r_k-r_i} & \ldots\ldots & a_{a+r_i,\,\beta+r_k} \end{vmatrix},$$

in which $\overline{U}=A+T$, $V=A-T$. By taking the inverse of both members, we have

$$(U')^{-1} A \overline{U}^{-1} = (V')^{-1} A \overline{V}^{-1},$$

whence $V'(U')^{-1} A \overline{U}^{-1} \overline{V} = A$, or $(U^{-1} V)' A (\overline{U}^{-1} \overline{V}) = A$. By comparison with (10) we see that $U^{-1} V$ or $(\overline{A} + \overline{T})^{-1} (\overline{A} - \overline{T})$ transforms H into itself.

On the other hand, if S is the matrix of any substitution which transforms H into itself, then a matrix T is uniquely defined by the formula $T=A(E-\overline{S})(E+\overline{S})^{-1}$, in which E is the unit matrix. The solution of this equation for S gives $S=(\overline{A}+\overline{T})^{-1}(\overline{A}-\overline{T})$.

Moreover, $\overline{T}' = (E+S')^{-1}(E-S')A = (E+S')^{-1}(E+S'-2S')A = A-2(E+S')_{-1}S'A$. But we have $T=A(2E-E-\overline{S})(E+\overline{S})^{-1} = 2A(E+\overline{S})^{-1} -A = 2A(E+A^{-1}S'^{-1}A)^{-1}-A$ (using (10))$= 2[(E+A^{-1}S'^{-1}A)A^{-1}]^{-1} -A = 2(A^{-1}+A^{-1}S'^{-})^{-1}-A = 2[A^{-1}S'^{-1}(S'+E)]^{-1}-A = 2(E+S')^{-1}S'A-A$. Hence $T+\overline{T}' = 0$.

This proof is given, in a somewhat different form, by A. Loewy in *Nova Acta der Kaiserl. Leop.-Carol. Deutschen Akademie der Naturforscher*, Vol. LXXI (1898), § 2.

$$S_{i,\,\mu+1} = \begin{vmatrix} 0 \dots\dots\dots 0 \\ \dots\dots\dots\dots\dots \\ 0 \dots\dots\dots 0 \\ a_{\alpha+r_i,\,\beta} \dots a_{\alpha+r_i,\,\beta+\mu-1} \end{vmatrix}, \quad S_{\mu+1,\,i} = \begin{vmatrix} 0 \dots\dots 0 \quad a_{\alpha,\,\beta+r_i} \\ 0 \dots\dots 0 \quad a_{\alpha+1,\,\beta+r_i} \\ \dots\dots\dots\dots\dots \\ 0 \dots\dots 0 \quad a_{\alpha+\mu-1,\,\beta+r_i} \end{vmatrix}$$

and $S_{i,\,\mu+2}$, $S_{\mu+2,\,i}$ have all elements zero. We observe in particular that when $e_i = e_k$, S_{ik} is a square matrix having zero elements above the second diagonal and hence but one non-vanishing element in the first row. If $e_i > e_k$, all the elements of the first row are zero. The non-zero elements satisfy certain conditions which it is not necessary to specify.

If now we assume $e_1 > e_2$, it follows from the remark just made concerning the first row of S_{ik} and from the conditions $e_2 \geqq e_3 \dots$ that all the elements of the first row of A must be zero except $a_{1,\,e_1}$. Hence the minor $A_{e_1,\,e_1}$ is zero and by applying transformation (II) we express H in the form

$$\eta_1 \bar{\eta}_1 - \eta_{e_1} \bar{\eta}_{e_1} + H_2,$$

in which H_2 contains only $n - 2$ variables, whose coefficients are the same as the minor of A obtained by erasing the first and the e_1-th rows and columns. If A_2, the matrix of the coefficients in H_2, contains but one non-vanishing element in its first row, it will be $a_{2,\,e_1-1}$. This happens when $e_1 - 1 > e_2$. We repeat the transformation (II) on H_2, obtaining

$$H_2 = \eta_2 \bar{\eta}_2 - \eta_{e_1-1} \bar{\eta}_{e_1-1} + H_4.$$

This process is to be continued as long as the first row of each successive matrix A, A_2, A_4, $\dots$ contains but one non-vanishing element. If $e_1 = e_2 + \varepsilon$ $(\varepsilon \gtreqless 0)$, the matrix $A_{2\varepsilon}$ contains the least two non-zero elements in its first row, namely $a_{\varepsilon+1,\,e_2}$ and $a_{\varepsilon+1,\,e_1+e_2}$. We reduce this second coefficient to zero by substituting for the variables * ζ_{e_2}, $\zeta_{e_1+e_2}$ in $H_{2\varepsilon}$ the expressions

$$\zeta_{e_2} = \eta_{e_2} + \bar{a}_{\varepsilon+1,\,e_1+e_2}\,\eta_{e_1+e_2}, \quad \zeta_{e_1+e_2} = -\,\bar{a}_{\varepsilon+1,\,e_2}\,\eta_{e_1+e_2}. \tag{12}$$

If other non-vanishing elements occur in the same row, they can be reduced to zero in like manner. It is to be observed that if any coefficient a_{ik} is changed by this transformation into b_{ik}, its conjugate a_{ki} is at the same time changed into $b_{ki} = \bar{b}_{ik}$.

We now continue with the transformation (II), it being necessary after each application to reduce all the elements in the first row of each remaining matrix

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* To avoid multiplicity of notation we shall repeatedly use the letters ζ and η to denote the variables before and after any transformation.

to zero except the one element which occurs in S_{11}. If e_1 is even, this process can be continued until all the rows and columns of A which pass through S_{11} have disappeared. If e_1 is odd, there will be one row and column left, containing the common element $a_{aa} (\alpha = \frac{1}{2}(e_1 + e_2))$. After reducing all the elements of this row and column to zero except a_{aa} by means of (12), we make the transformation (I) in which m is replaced by α, and $\varkappa$ runs from $\alpha + 1$ to n. One more term is thus obtained for the canonical form, the sign of which is unknown. In this manner we have replaced the first e_1 variables $\zeta_1, \ldots, \zeta_{e_1}$ of the form H by new variables which are reduced to the canonical form. The number of terms in this group which are certainly positive is $e_1/2$ or $(e_1-1)/2$ according as e_1 is even or odd. These numbers can be expressed by the single formula $\frac{1}{4}[2e_1 - 1 - (-1)^{e_1}]$.

We proceed to reduce the next set of e_2 variables $\zeta_{e_1+1}, \ldots, \zeta_{e_1+e_2}$ in the same way, and so on. The final result may be expressed in the following theorem :

If the characteristic equation $C = 0$ of the substitution S has an m-fold root ρ $(\rho \bar{\rho} = 1)$, and if the exponents of the corresponding elementary divisors are $e_1, e_2,$ $\ldots, (e_1 + e_2 + \ldots = m)$, then these numbers are restricted by the condition

$$\frac{1}{4} \sum_i [2e_i - 1 + (-1)^{e_i}] \leq p, \tag{13}$$

since the left member gives the number of terms which are certainly positive after H has been reduced to the canonical form.

This summation may extend to all the exponents e, since in case any of them are equal to unity the corresponding terms in (13) vanish.

We turn now to the case in which the m equal roots ρ are not unimodular. Let σ denote the root of $C = 0$ which satisfies the relation $\rho \bar{\sigma} = 1$, and let the exponents of the elementary divisors corresponding to this root be $e_1', e_2', \ldots,$ $(\Sigma e_i' = m', m'$ being the number of equal roots $\sigma)$. As will be seen presently, the numbers e_i' and e_i coincide. Suppose S reduced to the normal form

$$\begin{vmatrix} M_1 & & & & & \\ & \ddots & & & & \\ & & M_{\mu+1} & & & \\ & & & N_1 & & \\ & & & & \ddots & \\ & & & & & N_{\nu+2} \end{vmatrix}$$

in which M_i has the form indicated in (9), N_i is a similar matrix in σ of order e_i',

and $N_{\nu+2}$ is expressible in terms of the remaining roots of the characteristic equation.

The product $S'A\bar{S}$ will be made up of matrices obtained from $\mathbf{M}_i'A\overline{\mathbf{M}}_k$, $\mathbf{M}_i'A\overline{\mathbf{N}}_k$, $\mathbf{N}_i'A\overline{\mathbf{N}}_k$, etc., as described in connection with (11) except that in $\mathbf{M}_i'A\overline{\mathbf{M}}_k$, $\mathbf{N}_i'A\overline{\mathbf{N}}_k*$ every element is multiplied by $\rho\bar{\rho}$, $\sigma\bar{\sigma}$ respectively, and since these multipliers are not unity the elements all vanish when condition (10) is imposed. The same is true of all the matrices whose symbols contain the letter $\mathbf{N}_{\nu+2}$ excepting only $\mathbf{N}_{\nu+2}'A\overline{\mathbf{N}}_{\nu+2}$. Assume as before that the exponents e_i (and e_i') are arranged in descending order of magnitude.

Suppose now $e_1 > e_1'$. Then the matrices $\mathbf{M}_1'A\overline{\mathbf{N}}_k$ have all the elements of the first row zero and the determinant $S'A\bar{S}$ would vanish. Moreover, we can not have $e_1' > e_1$, since then the first column of $S'A\bar{S}$ would consist only of zeros. Hence $e_1' = e_1$. If $e_1 > e_i$ or e_i' $(i \gtreqless 2)$, the only non-zero element in the first row of A is $a_{1\beta}$, $\beta = m + e_1$. Otherwise there are additional non-zero elements in this row which we reduce to zero by means of the transformation (12). Supposing this has been done, we observe that the element $a_{\beta\beta}$ occurs in the matrix $\mathbf{N}_1'A\overline{\mathbf{N}}_1$, and is therefore zero. Hence in transformation (II), which we now apply, the term a_{mm} is zero. The form H then becomes

$$\eta_1\bar{\eta}_1 - \eta_\beta\bar{\eta}_\beta + H_2, \quad H_2 = \underset{\kappa,\lambda}{\Sigma} a_{\kappa\lambda}\eta_\kappa\bar{\eta}_\lambda \quad (\varkappa, \lambda \neq 1, \beta).$$

The determinant A_2 of H_2 is the minor of A obtained by striking out the first and the β-th rows and columns.

The above process is repeated on H_2, giving

$$H_2 = \eta_2\bar{\eta}_2 - \eta_{\beta-1}\bar{\eta}_{\beta-1} + H_4.$$

This may be continued until the variables with subscripts $1, 2, \ldots, e_1, \beta, \beta-1,$ $\ldots, \beta-e_1+1$ have been replaced by new variables in the canonical form

$$\overset{e_1}{\underset{i=1}{\Sigma}} \left(\eta_i\bar{\eta}_i - \eta_{\beta+1-i}\bar{\eta}_{\beta+1-i}\right).$$

The remaining portion of H, which we will denote by H_{2e_1}, has for the matrix of its coefficients the minor A_{2e_1} of A obtained by erasing all the rows and columns meeting in the matrices $\mathbf{M}_1'A\overline{\mathbf{N}}_1$, $\mathbf{N}_1'A\overline{\mathbf{M}}_1$.

We may now prove by a repetition of the argument used above that $e_2' = e_2$, since otherwise A_{2e_1} would vanish. We next reduce H_{2e_1} to a sum of e_2 positive

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* It will be convenient to use these symbols to denote the matrices of e_i rows and e_k columns formed by omitting the zeros.

and e_2 negative terms plus a form $H_{2(e_1+e_2)}$ the matrix of whose coefficients is the minor obtained from A_{2e_1} by erasing all the rows and columns meeting in the matrices $\mathbf{M}_2' A \, \overline{\mathbf{N}}_2$, $\mathbf{N}_2' A \, \overline{\mathbf{M}}_2$. This process is to be continued as long as there are e's left which are greater than unity. The coefficients of the remaining terms of H which are not affected by the preceding transformations form the matrix

$$\begin{vmatrix} 0 & \mathbf{M}_{\mu+1}' A \, \overline{\mathbf{N}}_{\mu+1} & 0 \\ \mathbf{N}_{\mu+1}' A \, \overline{\mathbf{M}}_{\mu+1} & 0 & 0 \\ 0 & 0 & \mathbf{N}_{\mu+2}' A \, \overline{\mathbf{N}}_{\mu+2} \end{vmatrix},$$

since by a repetition of a preceding argument we evidently have $\nu = \mu$. Now reduce all but one of the terms in the first row of this matrix to zero by means of (12) and then apply (II), and so proceed until all the elements of $\mathbf{M}_{\mu+1}' A \, \overline{\mathbf{N}}_{\mu+1}$ have disappeared.

We thus obtain an expression for H in which the first $2\,m$ variables have been reduced to the canonical form, the terms containing them being alternately positive and negative. There are m positive terms and hence in this case $m \lessgtr p$.

Suppose now that the characteristic equation $C = 0$ has m_1 equal roots ρ_1, m_2 equal roots ρ_2, ..., m_k equal roots ρ_k, none of these being unimodular and $m_1, m_2, \ldots, m_k \gtrless 0$; also the same number of equal roots $\sigma_1, \sigma_2, \ldots, \sigma_k$ such that $\rho_i \, \bar{\sigma}_i = 1$. Suppose further that there are μ_1 equal unimodular roots τ_1, μ_2 equal roots τ_2, ..., μ_κ equal roots τ_κ ($\mu_1, \ldots, \mu_\kappa \gtrless 1$), and let $e_{i1}, e_{i2}, \ldots$ denote the exponents of the elementary divisors corresponding to the root τ_i. If the substitution S be expressed by means of suitably chosen variables in the normal form corresponding to (9), it is evident that when H is also expressed in terms of the same variables it consists of groups of terms, the first group containing only the variables $\zeta_1, \ldots, \zeta_{2m_1}$, the next group containing only ζ_{2m_1+1}, ..., $\zeta_{2(m_1+m_2)}$, and so on to the $(k+1)$-th group containing only the variables $\zeta_a, \ldots, \zeta_{a+\mu_1}$ ($a = 2m_1 + \ldots + 2m_k$), etc. It is further clear that each group of terms can be reduced separately to the canonical form.* Counting up the total number of terms which are certainly positive we obtain, as a restriction on the possible number of equal roots, the formula

$$\sum_i m_i + \tfrac{1}{4} \sum_{a,\beta} \left[2\,e_{a\beta} - 1 + (-1)^{e_{a\beta}} \right] \lessgtr p, \quad \sum 2\,m_i + \sum e_{a\beta} = n.$$

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* The reduction of H to the canonical form is virtually contained in the processes and results of the *Acta Nova* memoir of Loewy previously quoted. It is believed that the different treatment presented above will be found advantageous in various respects.

§ 5.   *The Fixed Points when C Has Equal Roots.*

To determine the fixed points of a substitution whose characteristic equation has m equal roots ρ, consider S in the normal form (9). The fixed points corresponding to the root ρ are determined by the equations $\zeta_i' = \rho \zeta_i$. These reduce to

$$\zeta_2 = \zeta_3 = \ldots = \zeta_{e_1} = 0; \quad \zeta_{e_1+2} = \zeta_{e_1+3} = \ldots = \zeta_{e_1+e_2} = 0; \quad \ldots;$$
$$\zeta_k = 0 \quad (k = m+1, \; m+2, \ldots, n).$$

Each of these equations is equivalent to two equations in real variables. They therefore determine a flat space of

$$2(n-1) - 2\left[\Sigma(e_i - 1) + n - m\right] = 2\left[m - 1 - \Sigma(e_i - 1)\right]$$

dimensions. But since $\Sigma e_i = m$, this number reduces to $2(\lambda - 1)$, in which λ is the number of elementary divisors containing the multiple root ρ. From this result we deduce the evident generalization:

If $C = 0$ has m_1 equal roots ρ_1, m_2 equal roots ρ_2, $\ldots$, m_s equal roots ρ_s, then the fixed points of S constitue s flat spaces of dimensions $2(\lambda_1 - 1)$, $2(\lambda_2 - 1)$, $\ldots$, $2(\lambda_s - 1)$ respectively, in which λ_i is the number of elementary divisors containing the root ρ_i. None of these s loci have points in common.

This theorem includes the simple roots, since the corresponding values of λ are 1 and $2(\lambda - 1)$ has the required value zero for such cases.